

ON COMMUTATIVE p -SCHEMES OF ORDER p^4

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ABSTRACT. In this article, we consider the existence and schurity problem on commutative p -schemes of order p^4 . Using the thin radical and thin residue, we give sufficient conditions for such p -schemes to be schurian. We also give questions related to our results.

Key words: association scheme; p -scheme; schurian.

1. INTRODUCTION

An association scheme is a combinatorial object which is defined by some algebraic properties derived from a transitive permutation group (see Section 2 for definitions). So, we can regard association schemes as a generalization of groups. In this sense, p -schemes correspond to p -groups (see Section 2 for definitions). It is known that all p -schemes of order p are unique up to isomorphism. Unlike p -groups of order p^2 , the number of isomorphism classes of p -schemes of order p^2 is 3. The classification of p -schemes of order p^3 is far from being complete.

As we mentioned the above, every transitive permutation group G on a finite set X induces an association scheme \mathcal{R}_G , where \mathcal{R}_G is the set of orbits by the componentwise action of G on $X \times X$. We say that an association scheme S is *schurian* if $S = \mathcal{R}_G$ for a transitive permutation group G on X . Characterizing schurian association schemes is one of the major topics in the theory of association schemes. In the case of p -schemes, one can check that every p -scheme of order at most p^2 is schurian. In [2, 6], some non-schurian p -schemes of order p^3 are given.

For a given association scheme, we can define its thin radical $\mathbf{O}_\theta(S)$ and thin residue $\mathbf{O}^\theta(S)$, respectively (see Section 2 for definitions). We denote the orders of $\mathbf{O}_\theta(S)$ and $\mathbf{O}^\theta(S)$ by $n_{\mathbf{O}_\theta(S)}$ and $n_{\mathbf{O}^\theta(S)}$, respectively. According to [3], all of non-schurian commutative 2-schemes of order 16 are **as-16.no.55**, 59, 79, 85, 89, 90, 94, 95. Note that they satisfy $n_{\mathbf{O}_\theta(S)} = 2$ and $n_{\mathbf{O}^\theta(S)} = 8$. This article is concerned with commutative p -schemes of order p^4 , where p is an odd prime. We can construct a non-schurian commutative p -scheme of order p^4 with $n_{\mathbf{O}_\theta(S)} = p$ (see Example 3.1). Our motivation is to find non-schurian p -schemes except for the case $n_{\mathbf{O}_\theta(S)} = p$. Such attempt leads to classifying by schurian subclasses. Our main result is the following.

Theorem 1.1. *Let S be a commutative p -scheme of order p^4 . Assume that one of the following conditions holds.*

- (i) $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S)$, $n_{\mathbf{O}_\theta(S)} = p^2$ and $S//\mathbf{O}^\theta(S) \cong C_{p^2}$.
- (ii) $n_{\mathbf{O}_\theta(S)} = n_{\mathbf{O}^\theta(S)} = p^2$, $\mathbf{O}^\theta(S) \neq \mathbf{O}_\theta(S)$ and $n_s = p^2$ for each $s \in S \setminus \mathbf{O}_\theta(S)\mathbf{O}^\theta(S)$.
- (iii) $n_{\mathbf{O}_\theta(S)} = p^3$.

Then S is schurian.

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Note that we prove the above result in Theorems 3.1, 3.2 and 3.5.

This article is organized as follows. In Section 2, we prepare some terminology and notation. In Section 3, we give our main results. In Section 4, we prove Theorem 3.2.

2. PRELIMINARIES

Let X be a nonempty set, and let S be a partition of $X \times X$. The set S is called an *association scheme* (or shortly a *scheme*) on X if it satisfies the following conditions:

- (i) $1_X := \{(x, x) \mid x \in X\} \in S$;
- (ii) For each $s \in S$, $s^* := \{(x, y) \mid (y, x) \in s\} \in S$;
- (iii) For all $s, t, u \in S$ and $x, y \in X$, $a_{stu} := |xs \cap yt^*|$ is constant whenever $(x, y) \in u$, where $xs := \{y \in X \mid (x, y) \in s\}$.

For each s in S , we set $n_s := a_{ss^*1_X}$ and call this (positive) integer the *valency* of s . The unique relation containing a pair $(x, y) \in X \times X$ is denoted by $r(x, y)$. For a subset U of S , put $n_U := \sum_{u \in U} n_u$. We call n_U the *order* of U . The scheme S is called *p-valenced* if the valency of every element is a power of p , where p is a prime. In particular, a *p-valenced* scheme S is called a *p-scheme* if $|X|$ is also a power of p .

Let P and Q be nonempty subsets of S . We define PQ to be the set of all elements $s \in S$ such that there exist elements $p \in P$ and $q \in Q$ with $a_{pqs} \neq 0$. The set PQ is called the *complex product* of P and Q . If one of factors in a complex product consists of a single element s , then one usually writes s for $\{s\}$.

A nonempty subset T of S is called *closed* if $TT \subseteq T$. Note that a subset T of S is closed if and only if $\bigcup_{t \in T} t$ is an equivalence relation on X . A closed subset T is called *thin* if all elements of T have valency 1. The set $\{s \mid n_s = 1\}$ is called the *thin radical* of S and denoted by $\mathbf{O}_\theta(S)$. Note that T is thin if and only if T is a group under the relational product.

Let Y be a subset of X . For each $s \in S$, we define $s_Y := s \cap (Y \times Y)$. For each closed subset T of S , we set $T_Y := \{t_Y \mid t \in T\}$. Let x be an element in X , and T be a closed subset of S . Then T_{xT} is an association scheme on $xT := \bigcup_{t \in T} xt$, which is called *subscheme* of S defined by xT (see [9, Theorem 2.1.8]).

A closed subset T of S is called *strongly normal* in S , denoted by $T \triangleleft^\# S$, if $s^*Ts \subseteq T$ for every $s \in S$. We put $\mathbf{O}^\theta(S) := \bigcap_{T \triangleleft^\# S} T$ and call it the *thin residue* of S . Note that $\mathbf{O}^\theta(S) = \langle \bigcup_{s \in S} s^*s \rangle$ (see [8, Theorem 2.3.1]).

For each closed subset T of S , we define $X/T := \{xT \mid x \in X\}$ and $S//T := \{s^T \mid s \in S\}$, where $s^T := \{(xT, yT) \mid y \in xTsT\}$. Then $S//T$ is an association scheme on X/T , which is called the *quotient* (or *factor*) scheme of S over T (see [9, Theorem 4.1.3]). Note that $T \triangleleft^\# S$ if and only if $S//T$ is a group (see [8, Theorem 2.2.3]).

Let S_1 be an association scheme on X_1 . A bijective map ϕ from $X \cup S$ to $X_1 \cup S_1$ is called an *isomorphism* if it satisfies the following conditions:

- (i) $X^\phi \subseteq X_1$ and $S^\phi \subseteq S_1$;
- (ii) For all $x, y \in X$ and $s \in S$ with $(x, y) \in s$, $(x^\phi, y^\phi) \in s^\phi$.

An isomorphism ϕ from $X \cup S$ to $X \cup S$ is called an *automorphism* of S if $s^\phi = s$ for all $s \in S$. We denote by $\text{Aut}(S)$ the automorphism group of S . On the other hand, we say that S and S_1 are *algebraically isomorphic* or *have the same intersection numbers* if there exists a bijection ι from S to S_1 such that $a_{rst} = a_{r^\iota s^\iota t^\iota}$ for all $r, s, t \in S$.

Let F and H be association schemes on W and Y , respectively. For each $f \in F$ we define

$$\bar{f} := \{((w_1, y), (w_2, y)) \mid y \in Y, (w_1, w_2) \in f\}.$$

For each $h \in H \setminus \{1_Y\}$ we define

$$\bar{h} := \{((w_1, y_1), (w_2, y_2)) \mid w_1, w_2 \in W, (y_1, y_2) \in h\}.$$

Denote $F \wr H := \{\bar{f} \mid f \in F\} \cup \{\bar{h} \mid h \in H \setminus \{1_Y\}\}$. Then $F \wr H$ is an association scheme on $W \times Y$, which is called the *wreath product* of F and H . We note that if S is the wreath product of T_{xT} and $S//T$ for some closed subset T of S , then we simply denote S by $T \wr (S//T)$ instead of $T_{xT} \wr (S//T)$.

For each $s \in S$, we denote by σ_s the *adjacency matrix* of s . Namely σ_s is a matrix whose rows and columns are indexed by the elements of X and $(\sigma_s)_{xy} = 1$ if $(x, y) \in s$ and $(\sigma_s)_{xy} = 0$ otherwise.

We define the *left stabilizer* and *right stabilizer* of $s \in S$ by

$$L(s) = \{t \in S \mid ts = s\} \text{ and } R(s) = \{t \in S \mid st = s\}.$$

A map ϕ from a subset Y of X to X is called *faithful* if $r(x, y) = r(x^\phi, y^\phi)$ for $x, y \in Y$ (see [9]).

Remark 2.1. For any $x, y \in X$ there exists a faithful map ϕ from X to X such that $x^\phi = y$ if and only if $\text{Aut}(S)$ is transitive on X .

For $C, D \subseteq X$ and $\phi \in \text{Sym}(X)$, we say that C and D are *compatible with respect to ϕ* if

$$r(x, y) = r(x^\phi, y^\phi) \text{ for each } (x, y) \in C \times D.$$

We shall write $C \sim_\phi D$ if C and D are compatible with respect to ϕ , otherwise $C \not\sim_\phi D$.

The following lemma is a collection of basic facts.

Lemma 2.1 (See [1, 8]). *Let S be an association scheme on X . For $u, v, w \in S$, we have the following:*

- (i) $a_{u^*v}1_X = \delta_{u,v}n_u$;
- (ii) $a_{uvw} = a_{v^*u^*w^*}$;
- (iii) $a_{uvw^*}n_w = a_{vwu^*}n_u = a_{wuv^*}n_v$;
- (iv) $n_un_v = \sum_{s \in S} a_{uvs}n_s$;
- (v) $\text{lcm}(n_u, n_v) \mid a_{uvw}n_w$;
- (vi) $\text{gcd}(n_u, n_v) \geq |uv|$.

Theorem 2.2 (See Theorem B of [5]). *Assume that $\mathbf{O}^\theta(S) \subseteq \mathbf{O}_\theta(S)$ and that $\{s^*s \mid s \in S\}$ is linearly ordered with respect to set-theoretic inclusion. Then S is schurian.*

The following is well known.

Theorem 2.3. *Let S_1 and S_2 be association schemes. Then S_1 and S_2 are schurian if and only if $S_1 \wr S_2$ is schurian.*

Lemma 2.4 (See Lemma 3.3 of [2]). *Let S be a p -scheme of order p^3 such that $\mathbf{O}^\theta(S) \cong C_p \times C_p$. Then S is commutative if and only if $\mathbf{O}^\theta(S)s = s$ for each $s \in S \setminus \mathbf{O}^\theta(S)$.*

3. MAIN RESULTS

Throughout this section, we assume that S is a commutative p -scheme of order p^4 , where p is an odd prime. We divide our consideration into three subsections depending on $n_{\mathbf{O}_\theta(S)}$.

3.1. The case of $n_{\mathbf{O}_\theta(S)} = p$.

3.1.1. $n_{\mathbf{O}_\theta(S)} = p$ and $n_{\mathbf{O}^\theta(S)} = p$.

Since $\mathbf{O}^\theta(S)$ is a cyclic group, it follows from Theorem 2.2 that S is schurian.

3.1.2. $n_{\mathbf{O}_\theta(S)} = p$ and $n_{\mathbf{O}^\theta(S)} = p^3$.

In this case, we give a non-schurian example.

Example 3.1. Let T be a non-schurian commutative p -scheme of order p^3 such that $n_{\mathbf{O}_\theta(T)} = p$ and $n_{\mathbf{O}^\theta(T)} = p^2$ (see [2, Theorem 4.3]). Then $T \wr C_p$ is a non-schurian commutative p -scheme of order p^4 such that $n_{\mathbf{O}_\theta(T \wr C_p)} = p$ and $n_{\mathbf{O}^\theta(T \wr C_p)} = p^3$.

3.2. The case of $n_{\mathbf{O}_\theta(S)} = p^2$.

3.2.1. $n_{\mathbf{O}_\theta(S)} = p^2$ and $n_{\mathbf{O}^\theta(S)} = p$.

Since $C_p \cong \mathbf{O}^\theta(S) \subseteq \mathbf{O}_\theta(S)$, it follows from Theorem 2.2 that S is schurian.

3.2.2. $n_{\mathbf{O}_\theta(S)} = n_{\mathbf{O}^\theta(S)} = p^2$ and $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S)$.

In this case, S is isomorphic to a fission of the wreath product of two thin schemes (see [7, Theorem 1]).

Theorem 3.1. *Let S be a commutative p -scheme on X such that $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S)$, $n_{\mathbf{O}_\theta(S)} = p^2$ and $S//\mathbf{O}^\theta(S) \cong C_{p^2}$. Then S is schurian.*

Proof. If $\mathbf{O}^\theta(S) \cong C_{p^2}$, then it follows from Theorem 2.2 that S is schurian.

Suppose $\mathbf{O}^\theta(S) \cong C_p \times C_p$. Since $S//\mathbf{O}^\theta(S) \cong C_{p^2}$, there exists a unique closed subset T such that $\mathbf{O}^\theta(S) \subseteq T$ and $n_T = p^3$. By the definition of thin residue, we have $\mathbf{O}^\theta(T) \subseteq \mathbf{O}^\theta(S)$. We divide our consideration into two cases: $\mathbf{O}^\theta(T) = \mathbf{O}^\theta(S)$ and $\mathbf{O}^\theta(T) \neq \mathbf{O}^\theta(S)$.

(Case 1) $\mathbf{O}^\theta(T) = \mathbf{O}^\theta(S)$.

For $x \in X$, T_{xT} is a commutative p -scheme on xT of order p^3 such that $\mathbf{O}^\theta(T_{xT}) = \mathbf{O}_\theta(T_{xT}) \cong C_p \times C_p$. By Lemma 2.4, we have $n_t = p^2$ for each $t \in T \setminus \mathbf{O}^\theta(S)$. Note that T_{xT} is schurian.

In the rest of (Case 1), we shall show $n_s = p^2$ for each $s \in S \setminus T$.

Suppose that there exists $s \in S \setminus T$ such that $n_s = p$. If $|R(s)| = 1$, then $n_{\mathbf{O}_\theta(S)s} = p^3$ and $\sigma_s \sigma_s = p\sigma_{1_X} + \sum_{t \in T \setminus \mathbf{O}_\theta(S)} a_{s^*st} \sigma_t$. By Lemma 2.1(iv), there exist $t \in T \setminus \mathbf{O}_\theta(S)$ such that $a_{s^*st} \neq 0$. This contradicts $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S)$. Thus, we have $|R(s)| = p$.

Claim : $\sigma_s \sigma_s = p\sigma_{s'}$ for some $s' \in S$ with $n_{s'} = p$.

Let $s' \in ss$, $(\alpha, \beta) \in s'$ and $R(s) = \langle l_1 \rangle$. Then there exists $\gamma_1 \in \alpha s \cap \beta s^*$. Using $sl_1 = s$, for $(\alpha, \gamma_1) \in s$ we have $\gamma_2 \in \alpha s \setminus \{\gamma_1\}$ such that $\gamma_2 \in \alpha s \cap \gamma_1 l_1^*$. It follows from $l_1 s = s$ that $\beta \in \gamma_1 s = \gamma_2 l_1 s = \gamma_2 s$. Thus, we have $\gamma_2 \in \alpha s \cap \beta s^*$.

Since $sl_1^i = s$ for l_1^i ($2 \leq i \leq p-1$), by applying the same argument for l_1^i we obtain $|\alpha s \cap \beta s^*| = p$. This completes the proof of Claim.

By Lemma 2.1(iv) and Claim, we have $ss = s'$. Since $S//\mathbf{O}_\theta(S) \cong C_{p^2}$ and $s \notin \mathbf{O}_\theta(S)$, we have $s\mathbf{O}_\theta(S) \cap s'\mathbf{O}_\theta(S) = \emptyset$. Note that $R(s) = R(s')$.

By the argument used in the proof of Claim, we can show $\sigma_{s'} \sigma_s = p\sigma_{s''}$ for some $s'' \in S$ with $n_{s''} = p$. Note that $R(s) = R(s'')$ and $s'' = sss$. By repeating this process, we have $s^p \in T \setminus \mathbf{O}^\theta(S)$ with $n_{s^p} = p$, since $S//\mathbf{O}^\theta(S) \cong C_{p^2}$. But, this contradicts the fact that $n_t = p^2$ for each $t \in T \setminus \mathbf{O}^\theta(S)$.

Therefore, we have $n_s = p^2$ for each $s \in S \setminus \mathbf{O}^\theta(S)$, i.e., $S \cong \mathbf{O}^\theta(S) \wr C_{p^2}$. By Theorem 2.3, S is schurian.

(Case 2) $\mathbf{O}^\theta(T) \neq \mathbf{O}^\theta(S)$.

Since T is not thin and $\mathbf{O}^\theta(T) \subset \mathbf{O}^\theta(S)$, we have $n_{\mathbf{O}^\theta(T)} = p$. This implies $|\{t^*t \mid t \in T \setminus \mathbf{O}_\theta(S)\}| = 1$.

Suppose that there exists $s \in S \setminus T$ such that $n_s = p$. Then it is easy to see $|R(s)| = p$. By the same argument of (Case 1), we have $s^p \in T \setminus \mathbf{O}^\theta(S)$. This implies $|\{R(s) \mid s \in S \setminus \mathbf{O}_\theta(S)\}| = 1$. Thus, we have $n_{\mathbf{O}^\theta(S)} = p$, a contradiction.

Therefore, we have $n_s = p^2$ for each $s \in S \setminus T$. Since $s^*s = \mathbf{O}^\theta(S)$ and $t^*t = \mathbf{O}^\theta(T)$ for all $s \in S \setminus T$ and $t \in T \setminus \mathbf{O}^\theta(S)$, $\{s^*s \mid s \in S\}$ is linearly ordered. It follows from Theorem 2.2 that S is schurian. \square

The following example is a schurian commutative p -scheme such that $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S) \cong C_p \times C_p$ and $S//\mathbf{O}^\theta(S) \cong C_p \times C_p$.

Example 3.2. (See [4, Subsection 4.1]) There exists a schurian commutative p -scheme such that $\mathbf{O}^\theta(S) = \mathbf{O}_\theta(S) \cong C_p \times C_p$, $S//\mathbf{O}^\theta(S) \cong C_p \times C_p$ and $n_s = p$ for each $s \in S \setminus \mathbf{O}_\theta(S)$.

Question 3.1. Is there a non-schurian p -scheme algebraically isomorphic to Example 3.2 ?

3.2.3. $n_{\mathbf{O}_\theta(S)} = n_{\mathbf{O}^\theta(S)} = p^2$ and $\mathbf{O}^\theta(S) \neq \mathbf{O}_\theta(S)$.

Since $\mathbf{O}^\theta(S)$ has a nontrivial thin closed subset, we have $|\mathbf{O}^\theta(S) \cap \mathbf{O}_\theta(S)| = p$. By [9, Lemma 2.1.1], $\mathbf{O}^\theta(S)\mathbf{O}_\theta(S)$ is a closed subset. Also, we have $n_{\mathbf{O}^\theta(S)\mathbf{O}_\theta(S)} = p^3$.

Theorem 3.2. Let S be a commutative p -scheme on X such that $n_S = p^4$, $n_{\mathbf{O}_\theta(S)} = n_{\mathbf{O}^\theta(S)} = p^2$ and $\mathbf{O}^\theta(S) \neq \mathbf{O}_\theta(S)$. Assume $n_s = p^2$ for each $s \in S \setminus \mathbf{O}_\theta(S)\mathbf{O}^\theta(S)$. Then S is schurian.

In Section 4, we give our proof of Theorem 3.2.

Question 3.2. In Theorem 3.2, is it possible to exist $s \in S \setminus \mathbf{O}_\theta(S)\mathbf{O}^\theta(S)$ with $n_s = p$?

3.2.4. $n_{\mathbf{O}_\theta(S)} = p^2$ and $n_{\mathbf{O}^\theta(S)} = p^3$.

Lemma 3.3. Let S be a commutative p -scheme of order p^4 such that $n_{\mathbf{O}_\theta(S)} = p^2$ and $n_{\mathbf{O}^\theta(S)} = p^3$. Then $\mathbf{O}_\theta(S) \subseteq \mathbf{O}^\theta(S)$.

Proof. Suppose $\mathbf{O}_\theta(S) \not\subseteq \mathbf{O}^\theta(S)$. Since $\mathbf{O}^\theta(S)$ contains a nontrivial thin closed subset, we have $|\mathbf{O}^\theta(S) \cap \mathbf{O}_\theta(S)| = p$ and $\mathbf{O}^\theta(S)\mathbf{O}_\theta(S) = S$. The last equation implies $\mathbf{O}^\theta(S) = \langle s^*s \mid s \in \mathbf{O}^\theta(S)\mathbf{O}_\theta(S) \rangle = \langle s^*s \mid s \in \mathbf{O}^\theta(S) \rangle = \mathbf{O}^\theta(\mathbf{O}^\theta(S))$. This contradicts $\mathbf{O}^\theta(\mathbf{O}^\theta(S)) \subset \mathbf{O}^\theta(S)$ (see [8, Theorem 2.4.6]). \square

Proposition 3.4. Let S be a commutative p -scheme of order p^4 such that $n_{\mathbf{O}_\theta(S)} = p^2$, $n_{\mathbf{O}^\theta(S)} = p^3$ and $n_t = p^2$ for each $t \in \mathbf{O}^\theta(S) \setminus \mathbf{O}_\theta(S)$. Then we have $n_s \geq p^2$ for each $s \in S \setminus \mathbf{O}^\theta(S)$.

Proof. Suppose that there exists $s \in S \setminus \mathbf{O}^\theta(S)$ with $n_s = p$. Then $|R(s)| = 1$ or p .

When $|R(s)| = 1$, we have $n_{s\mathbf{O}_\theta(S)} = p^3$. Also, we have $\sigma_{s^*}\sigma_s = p\sigma_{1_X} + \sum_{t \in \mathbf{O}^\theta(S) \setminus \mathbf{O}_\theta(S)} a_{s^*st}\sigma_t$. Since every $t \in \mathbf{O}^\theta(S) \setminus \mathbf{O}_\theta(S)$ has the valency p^2 , this is impossible.

When $|R(s)| = p$, we have $n_{s\mathbf{O}_\theta(S)} = p^2$. Since $n_{s\mathbf{O}^\theta(S)} = p^3$ and $\mathbf{O}_\theta(S) \subseteq \mathbf{O}^\theta(S)$, we have $n_{s'} = p$ or p^2 for each $s' \in s\mathbf{O}^\theta(S) \setminus \{s\}$.

If $n_{s'} = p$, then it follows from the previous argument that $|R(s')| = p$.

If $n_{s'} = p^2$, then $|R(s')| \leq p^2$. Since $n_{s'\mathbf{O}^\theta(S)} = p^3$ and $n_{\mathbf{O}_\theta(S)} = p^2$, we have $|R(s')| \geq p$. Suppose $|R(s')| = p$. Then $s\mathbf{O}^\theta(S) = s'\mathbf{O}^\theta(S)$ but every element of $s'\mathbf{O}^\theta(S)$ has the valency p^2 , a contradiction. So, we have $|R(s')| = p^2$.

Thus, we have $s\mathbf{O}^\theta(S) = \bigcup_{s' \in s\mathbf{O}^\theta(S)} s'\mathbf{O}_\theta(S)$ and $n_{s'\mathbf{O}_\theta(S)} = p^2$ for each $s' \in s\mathbf{O}^\theta(S)$. This implies that $S//\mathbf{O}_\theta(S)$ must be a thin p -scheme of order p^2 . So, we have $\mathbf{O}^\theta(S) \subseteq \mathbf{O}_\theta(S)$, a contradiction.

Therefore, we have $n_s \geq p^2$ for each $s \in S \setminus \mathbf{O}^\theta(S)$. \square

Question 3.3. Is there a commutative p -scheme S of order p^4 such that $n_{\mathbf{O}_\theta(S)} = p^2$, $n_{\mathbf{O}^\theta(S)} = p^3$ and $n_s = p^2$ for each $s \in S \setminus \mathbf{O}_\theta(S)$?

3.3. The case of $n_{\mathbf{O}_\theta(S)} = p^3$.

In this subsection, we consider commutative p -schemes with $n_{\mathbf{O}_\theta(S)} = p^{n-1}$ and $n_S = p^n$ in general form. Note that $\mathbf{O}^\theta(S) \subseteq \mathbf{O}_\theta(S)$ since $S//\mathbf{O}_\theta(S) \cong C_p$.

Theorem 3.5. *Let S be a commutative p -scheme on X of order p^n such that $n_{\mathbf{O}_\theta(S)} = p^{n-1}$. Then S is schurian.*

Proof. We take $s \in S \setminus \mathbf{O}_\theta(S)$ such that $\min\{n_t \mid t \in S \setminus \mathbf{O}_\theta(S)\} = n_s$. Then $n_s = p^i$ for some $1 \leq i \leq n-1$. Note that $|R(s)| \leq p^i$ and $R(s) = L(s)$.

First of all, we show $|R(s)| = p^i$. Suppose $|R(s)| \leq p^{i-1}$. By Lemma 2.1(v), for each $t \in \mathbf{O}_\theta(S)$ st is a relation with valency p^i . Since $|\mathbf{O}_\theta(S)//R(s)| \geq p^{n-i}$, we have $n_{s\mathbf{O}_\theta(S)} \geq p^i \cdot p^{n-i} = p^n$, a contradiction.

Claim : $\sigma_s \sigma_s = n_s \sigma_{s'}$ for some $s' \in S$ with $n_{s'} = p^i$.

Let $(\alpha, \beta) \in s' \in ss$ and $l_1 \in R(s)$. Then there exists $\gamma_1 \in \alpha s \cap \beta s^*$. Since $sl_1 = s$, for $(\alpha, \gamma_1) \in s$, we have $\gamma_2 \in \alpha s \cap \gamma_1 l_1^*$ for some $\gamma_2 \in \alpha s \setminus \{\gamma_1\}$. It follows from $l_1 s = s$ that $\beta \in \gamma_1 s = \gamma_2 l_1 s = \gamma_2 s$. Thus, we have $\gamma_2 \in \alpha s \cap \beta s^*$.

Applying the same argument for each $l \in R(s) \setminus \{l_1\}$, we obtain $|\alpha s \cap \beta s^*| = p^i$. This completes the proof of Claim.

By Lemma 2.1(iv) and Claim, we have $ss = s'$. Since $S//\mathbf{O}_\theta(S) \cong C_p$ and $s \notin \mathbf{O}_\theta(S)$, we have $s\mathbf{O}_\theta(S) \cap s'\mathbf{O}_\theta(S) = \emptyset$. Note that $R(s) = R(s')$.

By the argument used in the proof of Claim, we can show $\sigma_{s'} \sigma_s = p^i \sigma_{s''}$ for some $s'' \in S$ with $n_{s''} = p^i$. By repeating this process, we have $S = \bigcup_{0 \leq j \leq p-1} s^j \mathbf{O}_\theta(S)$. This implies $\mathbf{O}^\theta(S) = s^* s$. Since $\{s^* s \mid s \in S\}$ is linearly ordered, it follows from Theorem 2.2 that S is schurian. \square

4. PROOF OF THEOREM 3.2

We denote $\mathbf{O}^\theta(S)\mathbf{O}_\theta(S)$ and $\mathbf{O}_\theta(S) \cap \mathbf{O}^\theta(S)$ by T and H , respectively. We put

$$I := \{0, 1, \dots, p^2 - 1\}, I^\circ := \{0, p, 2p, \dots, (p-1)p\}, I^\times := I \setminus I^\circ,$$

$$J := \{0, 1, \dots, p-1\} \text{ and } J^\times := J \setminus \{0\}.$$

Since $S//\mathbf{O}^\theta(S) \cong C_{p^2}$ or $C_p \times C_p$, without loss of generality, we can assume

$$S = \bigcup_{i,j \in J} \mathbf{O}^\theta(S) s_i t_j$$

such that $s_1 \in S \setminus T$, $t_1 \in \mathbf{O}_\theta(S) \setminus H$, $\mathbf{O}^\theta(S) s_1^i = \mathbf{O}^\theta(S) s_i$ and $\mathbf{O}^\theta(S) t_1^i = \mathbf{O}^\theta(S) t_i$ for each $i \in J$. Note that $T s_1^i = \bigcup_{j \in J} \mathbf{O}^\theta(S) s_i t_j$ for each $i \in J$.

From now on, we fix $\delta \in X$ and $r_0 \in \mathbf{O}^\theta(S) \setminus H$. We denote $\delta \mathbf{O}^\theta(S) s_1^i t_1^j$ by X_{i+jp} .

Remark 4.1. We have $T = \bigcup_{j \in J} \mathbf{O}^\theta(S) t_1^j = \bigcup_{j \in J} \mathbf{O}_\theta(S) r_0^j$.

For each $i \in J$, we consider the set of equivalence classes on $\bigcup_{j \in I^\circ} X_{i+j}$ induced by $\bigcup_{t \in \mathbf{O}_\theta(S)} t \cap (\bigcup_{j \in I^\circ} X_{i+j} \times \bigcup_{j \in I^\circ} X_{i+j})$. Denote it by $\{E_{i,j} \mid j \in J\}$.

For each $i \in J$, we take

$$\beta_{i,j} \in X_i \ (j \in J)$$

such that

$$\beta_{i,j} \in E_{i,j} \text{ and } \beta_{i,j+1} \in \beta_{i,j} r_0.$$

The subindex j of $\beta_{i,j}$ is reduced by modulo p . Note that $\bigcup_{l \in I^\circ} X_{i+l} = \{\beta_{i,j} t \mid j \in J, t \in \mathbf{O}_\theta(S)\}$.

Remark 4.2. $\{X_i \cap E_{i,j} \mid j \in J\}$ is the set of equivalence classes on X_i induced by $\bigcup_{h \in H} h \cap (X_i \times X_i)$.

Let $X_A := \bigcup_{i \in I} \{X_{(i,j)} \mid j \in J\}$ be a partition of X , where $\{X_{(i,j)} \mid j \in J\}$ is the set of equivalence classes on X_i induced by $\bigcup_{h \in H} h \cap (X_i \times X_i)$. The subindex j of $X_{(i,j)}$ is reduced by modulo p .

Remark 4.3. For $k, l \in J$, we have $\beta_{k,j} t_1^l \in X_{(k+lp,j)}$.

4.1. One-point stabilizer of the automorphism group.

In this subsection, we will show that for any $s \in S$, $\text{Aut}(S)_\delta$ is transitive on δs .

For a fixed $h_0 \in H \setminus \{1_X\}$, define $\phi : X \rightarrow X$ such that

(i) for each $i \in I^\circ$

$\phi|_{X_{(i,0)}}$ is the identity map,

(ii) for each $(i,j) \in (I^\circ \times J^\times) \bigcup (I^\times \times J)$

$\phi|_{X_{(i,j)}} : X_{(i,j)} \rightarrow X_{(i,j)}$ by $\alpha^\phi = \alpha h_0$.

Proposition 4.1. $\langle \phi \rangle$ is a nontrivial subgroup of $\text{Aut}(S)_\delta$ such that $\langle \phi \rangle$ is transitive on $X_{(i,j)}$ for each $(i,j) \in (I \times J) \setminus (I^\circ \times \{0\})$.

Proof. It follows from the definition of ϕ that $\langle \phi \rangle$ is transitive on $X_{(i,j)}$ for each $(i,j) \in (I \times J) \setminus (I^\circ \times \{0\})$. It suffices to verify $X_{(i,j)} \sim_\phi X_{(i',j')}$ for all $(i,j), (i',j') \in I \times J$.

For $(i,j), (i',j') \in I^\circ \times \{0\}$, we clearly have $X_{(i,j)} \sim_\phi X_{(i',j')}$ since $\phi|_{X_{(i,j)}}$ and $\phi|_{X_{(i',j')}} are the identity maps.$

For $(i,j), (i',j') \in (I \times J) \setminus (I^\circ \times \{0\})$, we have

$$\begin{aligned} r(x^\phi, y^\phi) &= r(xh_0, yh_0) = r(xh_0, x)r(x, y)r(y, yh_0) \\ &= h_0^* r(x, y) h_0 = r(x, y) \end{aligned}$$

for each $(x, y) \in X_{(i,j)} \times X_{(i,j)}$.

For $(i,j) \in I^\circ \times \{0\}$ and $(i',j') \in (I \times J) \setminus (I^\circ \times \{0\})$, we divide our consideration into three cases.

(Case 1) $i = i'$ and $j \neq j'$: we have $X_{(i,j)} \sim_\phi X_{(i',j')}$ since $X_{(i,j)} \times X_{(i',j')} \subseteq t$ for some $t \in \mathbf{O}^\theta(S) \setminus H$.

(Case 2) $i \not\equiv i' \pmod{p}$: Then either $j = j'$ or $j \neq j'$. Whichever the case may be, we have $X_{(i,j)} \times X_{(i',j')} \subseteq s$ for some $s \in S \setminus T$. So, $X_{(i,j)} \sim_\phi X_{(i',j')}$.

(Case 3) $i \neq i'$, $i \equiv i' \pmod{p}$ and $j \neq j'$: we have $X_{(i,j)} \sim_\phi X_{(i',j')}$ since $X_{(i,j)} \times X_{(i',j')} \subseteq t$ for some $t \in T \setminus \mathbf{O}_\theta(S)$.

□

Define $\psi : X \rightarrow X$ such that

(i) for each $i \in I^\circ$ and $j \in J$

$$\psi|_{X_{(i,j)}} = \phi|_{X_{(i,j)}},$$

(ii) for each $i \in J^\times$ and $l, j \in J$

$$\psi|_{X_{(i+lp,j)}} : X_{(i+lp,j)} \rightarrow X_{(i+lp,j+1)} \text{ by } (\beta_{i,j} t_1^l h)^\psi = \beta_{i,j+1} t_1^l h \text{ (} h \in H \text{)}.$$

Proposition 4.2. $\langle \psi \rangle$ is a nontrivial subgroup of $\text{Aut}(S)_\delta$ such that for each $(l, i) \in I^\circ \times J^\times$, $\langle \psi \rangle$ is transitive on $\{X_{(i+l,j)} \mid j \in J\}$.

Proof. First of all, we prove that every pair of subsets in $\{X_{(i,j)} \mid (i,j) \in I^\circ \times J\} \cup \{X_i \mid i \in I^\times\}$ is compatible with respect to ψ .

Since $\psi|_{X_{(i,j)}} = \phi|_{X_{(i,j)}}$ for each $(i,j) \in I^\circ \times J$, it follows from the proof of Proposition 4.1 that $X_{(i,j)} \sim_\psi X_{(i',j')}$ for $(i,j), (i',j') \in I^\circ \times J$.

For $(i,j) \in I^\circ \times J$ and $i' \in I^\times$, we have $X_{(i,j)} \sim_\psi X_{i'}$ since $X_{(i,j)} \times X_{i'} \subseteq s$ for some $s \in S \setminus T$.

For $i, i' \in I^\times$ ($i \not\equiv i' \pmod{p}$), we have $X_i \sim_\psi X_{i'}$ since $X_i \times X_{i'} \subseteq s$ for some $s \in S \setminus T$.

For $i, i' \in I^\times$ ($i \equiv i' \pmod{p}$), let us put $i = i_1 + l$ and $i' = i_1 + l'$ ($i_1 \in J$ and $l, l' \in I^\circ$), we show $X_{i_1+l} \sim_\psi X_{i_1+l'}$. Let $(x, y) \in X_{i_1+l} \times X_{i_1+l'}$. Then $x = \beta_{i_1,j} t_1^l h_1$ and $y = \beta_{i_1,j'} t_1^{l'} h_2$ for some $j, j' \in J^\times$ and $h_1, h_2 \in H$. We have

$$\begin{aligned} r(x, y) &= r(\beta_{i_1,j} t_1^l h_1, \beta_{i_1,j'}) r(\beta_{i_1,j}, \beta_{i_1,j'}) r(\beta_{i_1,j'}, \beta_{i_1,j'} t_1^{l'} h_2) \\ &= (t_1^l h_1)^* r(\beta_{i_1,j}, \beta_{i_1,j'}) t_1^{l'} h_2 \end{aligned}$$

and

$$\begin{aligned} r(x^\psi, y^\psi) &= r(\beta_{i_1,j+1} t_1^l h_1, \beta_{i_1,j+1}) r(\beta_{i_1,j+1}, \beta_{i_1,j'+1}) r(\beta_{i_1,j'+1}, \beta_{i_1,j'+1} t_1^{l'} h_2) \\ &= (t_1^l h_1)^* r(\beta_{i_1,j+1}, \beta_{i_1,j'+1}) t_1^{l'} h_2. \end{aligned}$$

(Case 1) $j = j'$: we have $r(x^\psi, y^\psi) = h_1^* (t_1^l)^* t_1^{l'} h_2 = r(x, y)$.

(Case 2) $j \neq j'$: we have $r(x^\psi, y^\psi) = h_1^* (t_1^l)^* r_0^m t_1^{l'} h_2 = r(x, y)$ for some $m \in J^\times$.

Finally, it follows from the definition of ψ that $\langle \psi \rangle$ is transitive on $\{X_{(i+l,j)} \mid j \in J\}$. \square

It follows from Propositions 4.1 and 4.2 that for each $s \in S$, $\langle \phi, \psi \rangle$ is transitive on δs .

4.2. Transitivity of the automorphism group.

In this subsection, for all $\alpha, \beta \in X$ we will construct a faithful map $\phi : X \rightarrow X$ such that $\alpha^\phi = \beta$. We divide our consideration into four cases :

- (I) $\alpha H = \beta H$,
- (II) $\alpha H \neq \beta H$ and $\alpha \mathbf{O}^\theta(S) = \beta \mathbf{O}^\theta(S)$,
- (III) $\alpha \mathbf{O}^\theta(S) \neq \beta \mathbf{O}^\theta(S)$ and $\alpha T = \beta T$,
- (IV) $\alpha T \neq \beta T$.

In the case (I), without loss of generality, we assume $\alpha, \beta \in X_{(0,j)}$ for some j .

Put $\alpha_0 = \alpha$ and $\beta_0 = \beta$, and take $\alpha_i, \beta_i \in X$ such that $\alpha_{i+1} = \alpha_i t_1$ and $\beta_{i+1} = \beta_i t_1$ for each $i \in J \setminus \{p-1\}$.

Define $\phi : X \rightarrow X$ such that

(i) for each $l \in I^\circ$

$$\phi|_{X_{(l,j)}} : X_{(l,j)} \rightarrow X_{(l,j)} \text{ by } \alpha_{\frac{l}{p}} h \mapsto \beta_{\frac{l}{p}} h \text{ (} h \in H \text{)},$$

(ii)

$\phi|_{X \setminus \bigcup_{l \in I^\circ} X_{(l,j)}}$ is the identity map.

Proposition 4.3. For $\alpha, \beta \in X$ ($\alpha H = \beta H$), $\phi : X \rightarrow X$ is a faithful map such that $\alpha^\phi = \beta$.

Proof. Set $X_B = \{X_{(l,j)} \mid l \in I^\circ\}$. We divide X_A into two parts, i.e., X_B and $X_A \setminus X_B$.

For $X_{(i,j)}, X_{(i',j')} \in X_A \setminus X_B$, we clearly have $X_{(i,j)} \sim_\phi X_{(i',j')}$ since $\phi|_{X \setminus \bigcup_{l \in I^\circ} X_{(l,j)}}$ is the identity map.

For $X_{(l,j)} \in X_B$ and $X_{(i,k)} \in X_A \setminus X_B$,

(Case 1) $i \in I^\circ$: we have $X_{(l,j)} \sim_\phi X_{(i,k)}$ since $X_{(l,j)} \times X_{(i,k)} \subseteq s$ for some $s \in T \setminus \mathbf{O}_\theta(S)$.

(Case 2) $i \in I^\times$: we have $X_{(l,j)} \sim_\phi X_{(i,k)}$ since $X_{(l,j)} \times X_{(i,k)} \subseteq s$ for some $s \in S \setminus T$.

For $X_{(l,j)}, X_{(l',j')} \in X_B$, let us take $(x, y) \in X_{(l,j)} \times X_{(l',j')}$. Then $x = \alpha_{\frac{l}{p}} h_1$ and $y = \alpha_{\frac{l'}{p}} h_2$ for some $h_1, h_2 \in H$. We have

$$r(x, y) = r(\alpha_{\frac{l}{p}} h_1, \alpha_{\frac{l}{p}} h_1) r(\alpha_{\frac{l}{p}}, \alpha_{\frac{l'}{p}}) r(\alpha_{\frac{l'}{p}}, \alpha_{\frac{l'}{p}} h_2)$$

and

$$r(x^\phi, y^\phi) = r(\beta_{\frac{l}{p}} h_1, \beta_{\frac{l}{p}} h_1) r(\beta_{\frac{l}{p}}, \beta_{\frac{l'}{p}}) r(\beta_{\frac{l'}{p}}, \beta_{\frac{l'}{p}} h_2).$$

(Case 1) $l = l'$: we have $r(x^\phi, y^\phi) = h_1^* h_2 = r(x, y)$.

(Case 2) $l \neq l'$: we have $r(x^\phi, y^\phi) = h_1^* t_1^m h_2 = r(x, y)$ for some $m \in J^\times$. \square

In the case (II), without loss of generality, we assume $\alpha, \beta \in X_0$ such that $\alpha \in X_{(0,0)}$ and $\beta \in X_{(0,1)}$. By Proposition 4.3, we can assume $\alpha = \beta_{0,0}$ and $\beta = \beta_{0,1}$.

Define $\phi : X \rightarrow X$ such that

(i) for each $i, j \in J$

$$\phi|_{X_{(ip,j)}} : X_{(ip,j)} \rightarrow X_{(ip,j+1)} \text{ by } (\beta_{0,j} t_1^i h)^\phi = \beta_{0,j+1} t_1^i h \ (h \in H),$$

(ii)

$\phi|_{\bigcup_{i \in I^\times} X_i}$ is the identity map.

Proposition 4.4. For $\alpha, \beta \in X$ ($\alpha H \neq \beta H$ and $\alpha \mathbf{O}^\theta(S) = \beta \mathbf{O}^\theta(S)$), $\phi : X \rightarrow X$ is a faithful map such that $\alpha^\phi = \beta$.

Proof. For $i, i' \in I^\times$, we clearly have $X_i \sim_\phi X_{i'}$ since $\phi|_{\bigcup_{i \in I^\times} X_i}$ is the identity map.

For $i \in I^\circ$ and $i' \in I^\times$, we have $X_i \sim_\phi X_{i'}$ since $X_i \times X_{i'} \subseteq s$ for some $s \in S \setminus T$.

For $ip, i'p \in I^\circ$, let us take $(x, y) \in X_{ip} \times X_{i'p}$. Then $x = \beta_{0,j} t_1^i h_1$ and $y = \beta_{0,j'} t_1^{i'} h_2$ for some $j, j' \in J$ and $h_1, h_2 \in H$. We have

$$r(x, y) = r(\beta_{0,j} t_1^i h_1, \beta_{0,j} h_1) r(\beta_{0,j}, \beta_{0,j'}) r(\beta_{0,j'}, \beta_{0,j'} t_1^{i'} h_2)$$

and

$$r(x^\phi, y^\phi) = r(\beta_{0,j+1} t_1^i h_1, \beta_{0,j+1} h_1) r(\beta_{0,j+1}, \beta_{0,j'+1}) r(\beta_{0,j'+1}, \beta_{0,j'+1} t_1^{i'} h_2).$$

Since $r(\beta_{0,j}, \beta_{0,j'}) = r(\beta_{0,j+1}, \beta_{0,j'+1})$, we have $r(x^\phi, y^\phi) = r(x, y)$. \square

In the case (III), without loss of generality, we assume $\alpha \in X_0, \beta \in X_p$. By Propositions 4.3 and 4.4, we can assume $\alpha = \beta_{0,0}$ and $\beta = \beta_{0,0}t_1$.

Define $\phi : X \rightarrow X$ such that for each $i, j \in J$

$$\phi|_{\bigcup_{l \in I^\circ} X_{(i+l,j)}} : \bigcup_{l \in I^\circ} X_{(i+l,j)} \rightarrow \bigcup_{l \in I^\circ} X_{(i+l,j)} \text{ by } (\beta_{i,j}t)^\phi = \beta_{i,j}t_1t \ (t \in \mathbf{O}_\theta(S)).$$

Proposition 4.5. *For $\alpha, \beta \in X$ ($\alpha \mathbf{O}^\theta(S) \neq \beta \mathbf{O}^\theta(S)$ and $\alpha T = \beta T$), $\phi : X \rightarrow X$ is a faithful map such that $\alpha^\phi = \beta$.*

Proof. For $i, i' \in J$, let us take $(x, y) \in X_i \times X_{i'}$. Then $x = \beta_{i,j}l_1$ and $y = \beta_{i',j'}l_2$ for some $j, j' \in J$ and $l_1, l_2 \in \mathbf{O}_\theta(S)$. We have

$$r(x, y) = r(\beta_{i,j}l_1, \beta_{i,j})r(\beta_{i,j}, \beta_{i',j'})r(\beta_{i',j'}, \beta_{i',j'}l_2)$$

and

$$r(x^\phi, y^\phi) = r(\beta_{i,j}t_1l_1, \beta_{i,j})r(\beta_{i,j}, \beta_{i',j'})r(\beta_{i',j'}, \beta_{i',j'}t_1l_2).$$

(Case 1) $i \neq i'$: we have $r(x^\phi, y^\phi) = l_1^*t_1s_1^kt_1l_2 = l_1^*s_1^kl_2 = r(x, y)$ for some $k \in J^\times$.

(Case 2) $i = i'$ and $j \neq j'$: we have $r(x^\phi, y^\phi) = l_1^*t_1^*r_0^kt_1l_2 = l_1^*r_0^kl_2 = r(x, y)$ for some $k \in J^\times$.

(Case 3) $i = i'$ and $j = j'$: we have $r(x^\phi, y^\phi) = l_1^*l_2 = r(x, y)$. □

In the case (IV), without loss of generality, we assume $\alpha \in X_0, \beta \in X_1$. By Propositions 4.3, 4.4 and 4.5, we can assume $\alpha = \beta_{0,0} \in X_{(0,0)}$ and $\beta = \beta_{1,0} \in X_{(1,0)}$.

Define $\phi : X \rightarrow X$ such that for each $i \in J$

$$\phi|_{\bigcup_{l \in I^\circ} X_{i+l}} : \bigcup_{l \in I^\circ} X_{i+l} \rightarrow \bigcup_{l \in I^\circ} X_{i+1+l} \text{ by } (\beta_{i,j}t)^\phi = \beta_{i+1,j}t \ (j \in J, t \in \mathbf{O}_\theta(S)).$$

Proposition 4.6. *For $\alpha, \beta \in X$ ($\alpha T \neq \beta T$), $\phi : X \rightarrow X$ is a faithful map such that $\alpha^\phi = \beta$.*

Proof. For $i, i' \in J$, let us take $(x, y) \in X_i \times X_{i'}$. Then $x = \beta_{i,j}l_1$ and $y = \beta_{i',j'}l_2$ for some $j, j' \in J$ and $l_1, l_2 \in \mathbf{O}_\theta(S)$. We have

$$r(x, y) = r(\beta_{i,j}l_1, \beta_{i,j})r(\beta_{i,j}, \beta_{i',j'})r(\beta_{i',j'}, \beta_{i',j'}l_2)$$

and

$$r(x^\phi, y^\phi) = r(\beta_{i+1,j}l_1, \beta_{i+1,j})r(\beta_{i+1,j}, \beta_{i'+1,j'})r(\beta_{i'+1,j'}, \beta_{i'+1,j'}l_2).$$

(Case 1) $i \neq i'$: we have $r(x^\phi, y^\phi) = l_1^*s_1^kl_2 = r(x, y)$ for some $k \in J^\times$.

(Case 2) $i = i'$ and $j \neq j'$: we have $r(x^\phi, y^\phi) = l_1^*r_0^kl_2 = r(x, y)$ for some $k \in J^\times$.

(Case 3) $i = i'$ and $j = j'$: we have $r(x^\phi, y^\phi) = l_1^*l_2 = r(x, y)$. □

It follows from Propositions 4.3 – 4.6 that $\text{Aut}(S)$ is transitive on X .

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